METHODS OF QUALITATIVE THEORY OF CELLULAR NONLINEAR NETWORKS

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Outline

1. Cellular Nonlinear Networks (CNNs).
   (a) Mathematical model.
   (b) Dynamic behavior: analytical and numerical techniques.
      i. Equilibrium points and stability.
      ii. Periodic attractors.
      iii. Non-periodic attractors (Tori, Strange and/or Chaotic attractors).
2. Conclusions

CNN mathematical model

$$
\begin{aligned}
L(D_t)x_{ij}(t) \quad &= \sum_{|n|\le r, |m|\le r} T^A_{mn}\, f(x_{i+m,j+n}) \;+\; T^C_{mn}\, x_{i+m,j+n} \\
&+ \sum_{|n|\le r, |m|\le r} T^B_{mn}\, u_{i+m,j+n} + I
\end{aligned}
$$

$$
\boldsymbol{T^{A,B,C}} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \qquad (1 \le i \le N, \quad 1 \le j \le M)
$$

Inputs and bias are assumed to be constant (autonomous systems)

$x_{ij}(t)$: state variable

$L(D_t)$: rational function of the operator $D_t = \frac{d}{dt}$

$f(\cdot)$: nonlinear memoryless output function

$u_{ij}(t)$: input terms

I: bias term

Initial Conditions $[x_{ij}(0)]$ & *Boundary conditions*

$$\Downarrow$$

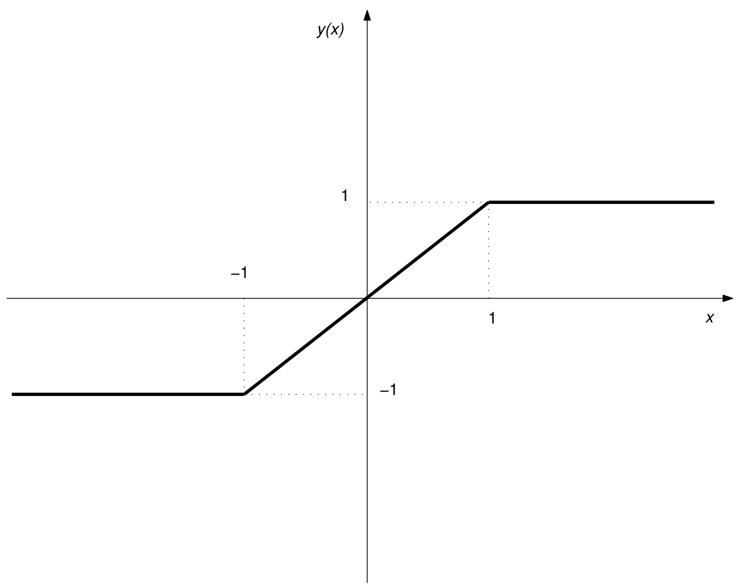
Identical cells, containing only one nonlinear memoryless element.

Local connections, described by space-invariant templates.

Chua-Yang CNN mathematical model

$$\frac{dx_{ij}}{dt} = -x_{ij} + \sum_{|m|\leq r, |n|\leq r} T^A_{mn}\, y(x_{i+m,j+n})$$

$$+ \sum_{|n|\leq r, |n|\leq r} T^B_{mn}\, u_{i+m,j+n} + \hat{I}$$



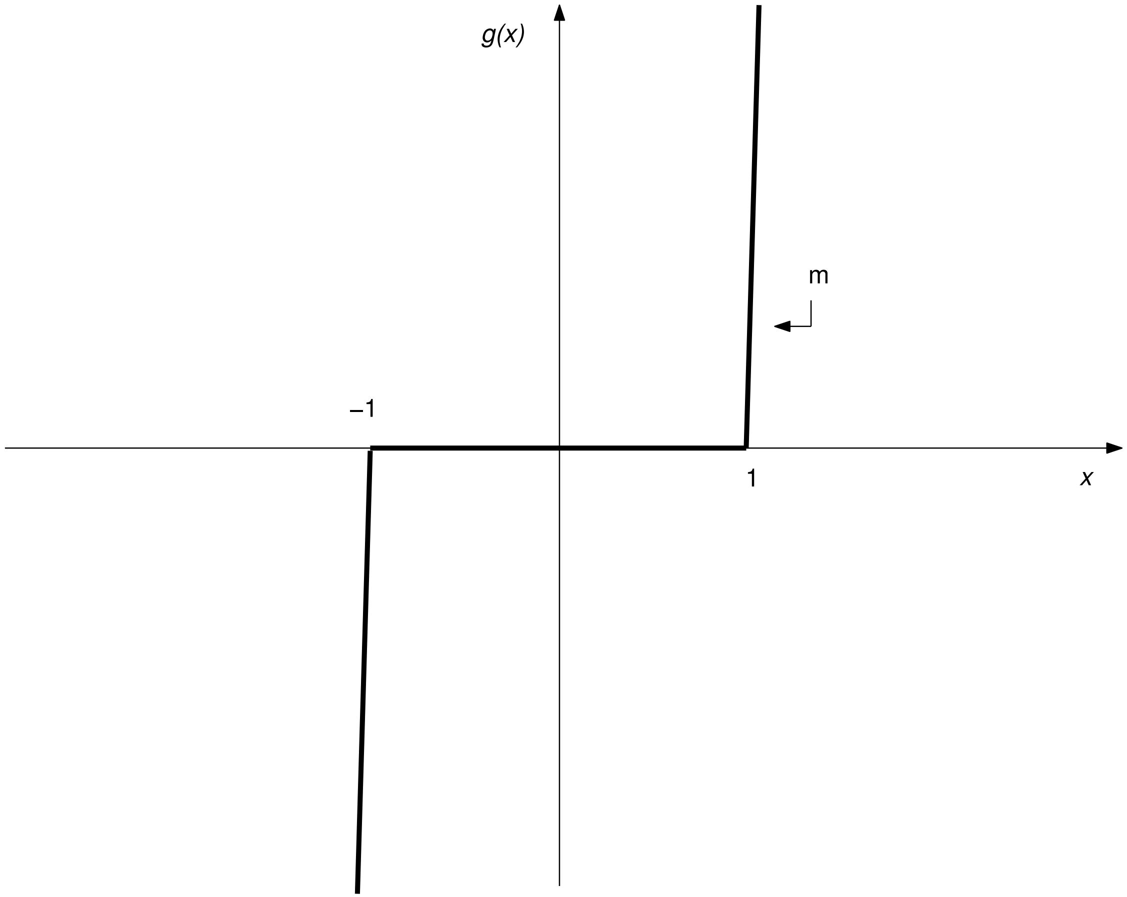
Full Range CNN model

$$\dot{x}_{ij} \;=\; -\, x_{ij} + \sum_{|m|\le r, |n|\le r} T^{A}_{mn}\, y(x_{i+m,j+n}) - \; g(x_{ij})$$

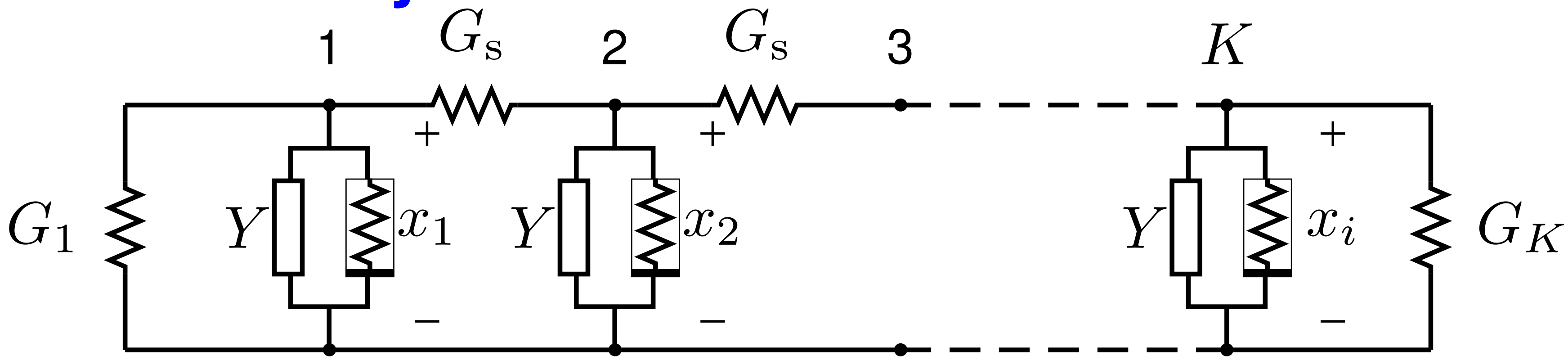
$$+ \sum_{|m|\le r, |n|\le r} T^{B}_{mn}\, u_{i+m,j+n} + \hat{I}$$

$$g(x_{ij}) = m[x_{ij} - y(x_{i,j})] = \begin{cases} m(x_{ij}+1) & x_{ij} < -1 \\ 0 & |x_{ij}| \le 1 \\ m(x_{ij}-1) & x_{ij} > 1 \end{cases}$$

Full Range CNN nonlinear function (cont.)



Nonlinear arrays of oscillators



$$Y(D_t)x_i(t) = -f[x_i(t)] - 2G_s x_i(t) + G_s x_{i+1}(t) + G_s x_{i-1}(t)$$

* $L(D_t) = Y(D_t)$
* $f(\cdot)$: voltage-controlled characteristic of the nonlinear resistor.
* $T_m^A = [0 \ \ -1 \ \ 0]$
* $T_m^C = [G_s \ \ -2G_s \ \ G_s]$

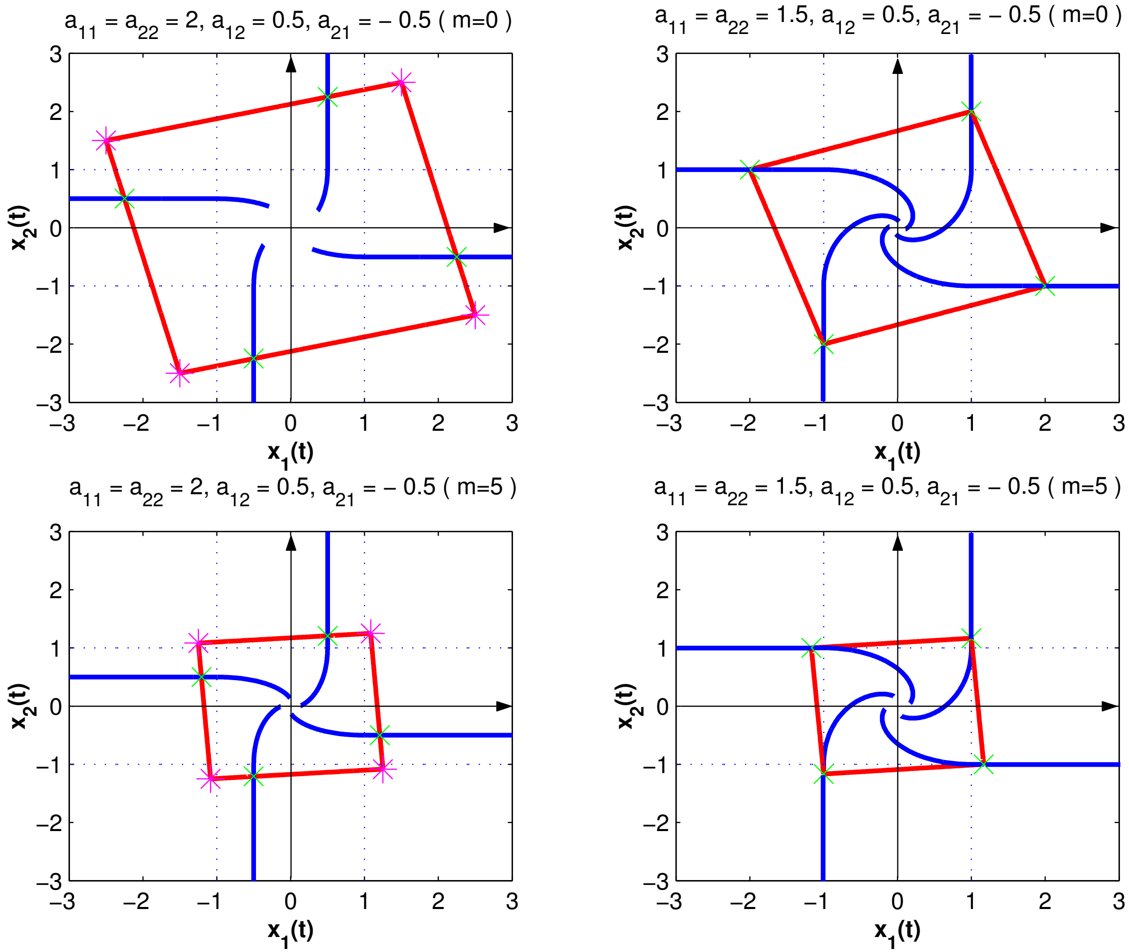
Dynamic behavior: Equilibrium points

1. Determination of all the possible stable and unstable equilibrium points.
2. Equilibrium point bifurcations.
   (a) Local bifurcations (saddle-node, pitchfork, Hopf bifurcations).
   (b) Global bifurcations (homoclinic and heteroclinic bifurcations).
3. Domains of attractions of the stable equilibrium points.
4. Conditions for the existence of at least one stable equilibrium point.
   (a) The existence of at least one stable equilibrium point does not imply stability.
   (b) The absence of stable equilibrium points implies instability.

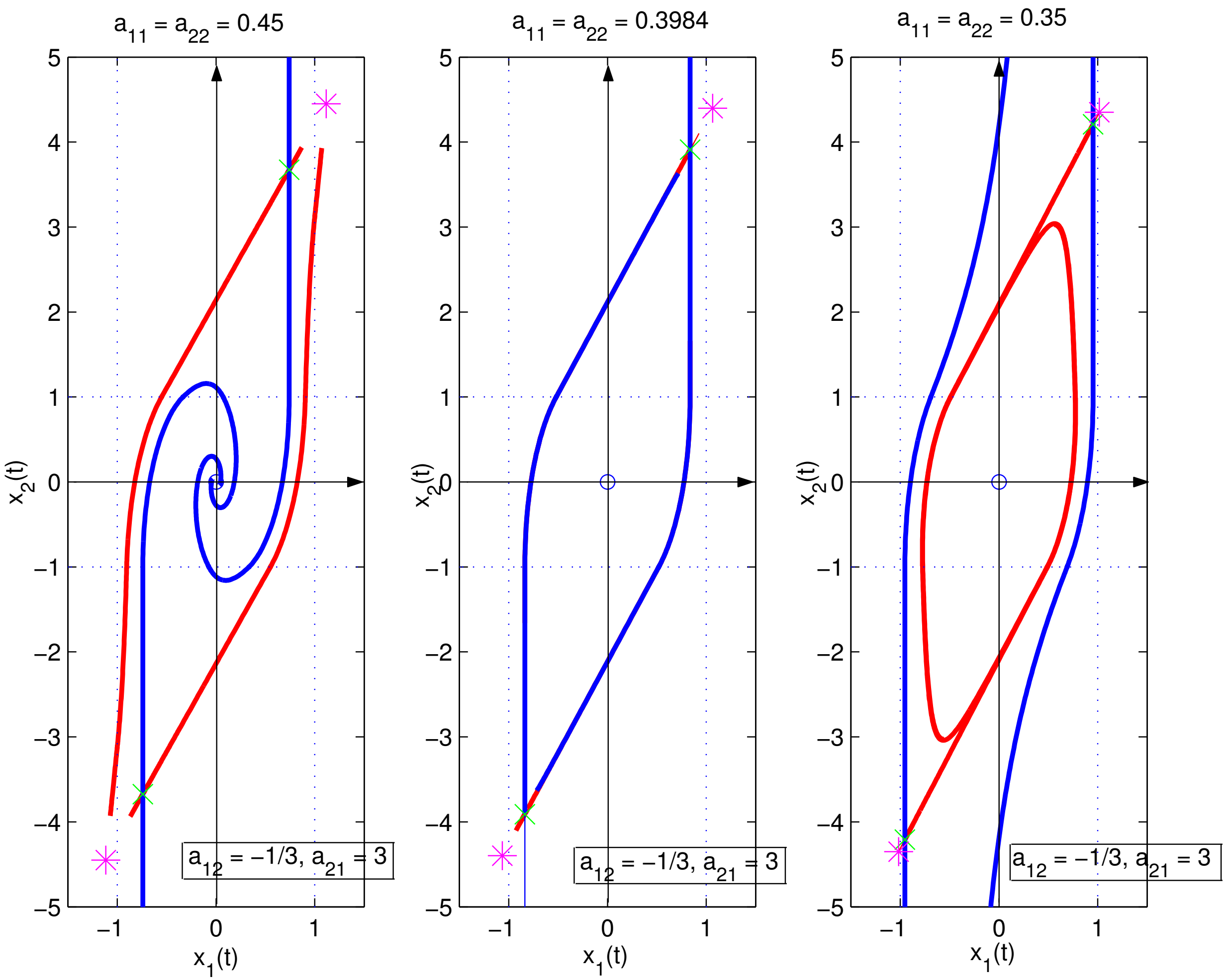
1. A network composed by $N \times M$ cells exhibits $3^{N \times M}$ linear regions.
   (a) Central region: all the cell lye in the linear part of their characteristics.
   (b) Saturation regions: all the cells are saturated to ± 1.
   (c) Partial saturation regions: only some cells are saturated.
2. Each region may present at most one equilibrium point.
3. If the central element of the template satisfies $T_{00}^{A} > 1$, then
   (a) If a saturation region presents an equilibrium point, then it is stable.
   (b) If a partial saturation region (or the central region) presents an equilibrium point, then it is unstable.

All the algorithms for equilibrium point computation in a CNN have an exponential complexity

Equilibrium point local bifurcations in 2 cell CYCNNs and FRCNNs (cont.)

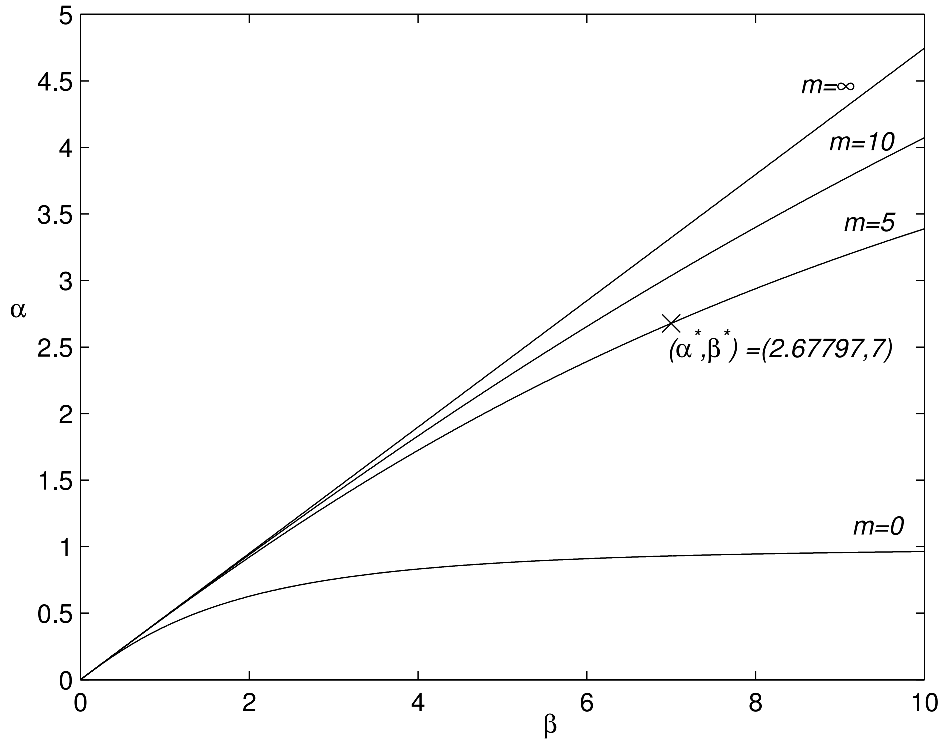


Stable and **Unstable** manifolds.



Intersection between **Stable** *and* **Unstable** *manifolds.*

There exist sets of **identical parameters** *for which CYCNN and FRCNN models exhibit a* **qualitatively different dynamic behavior***.*

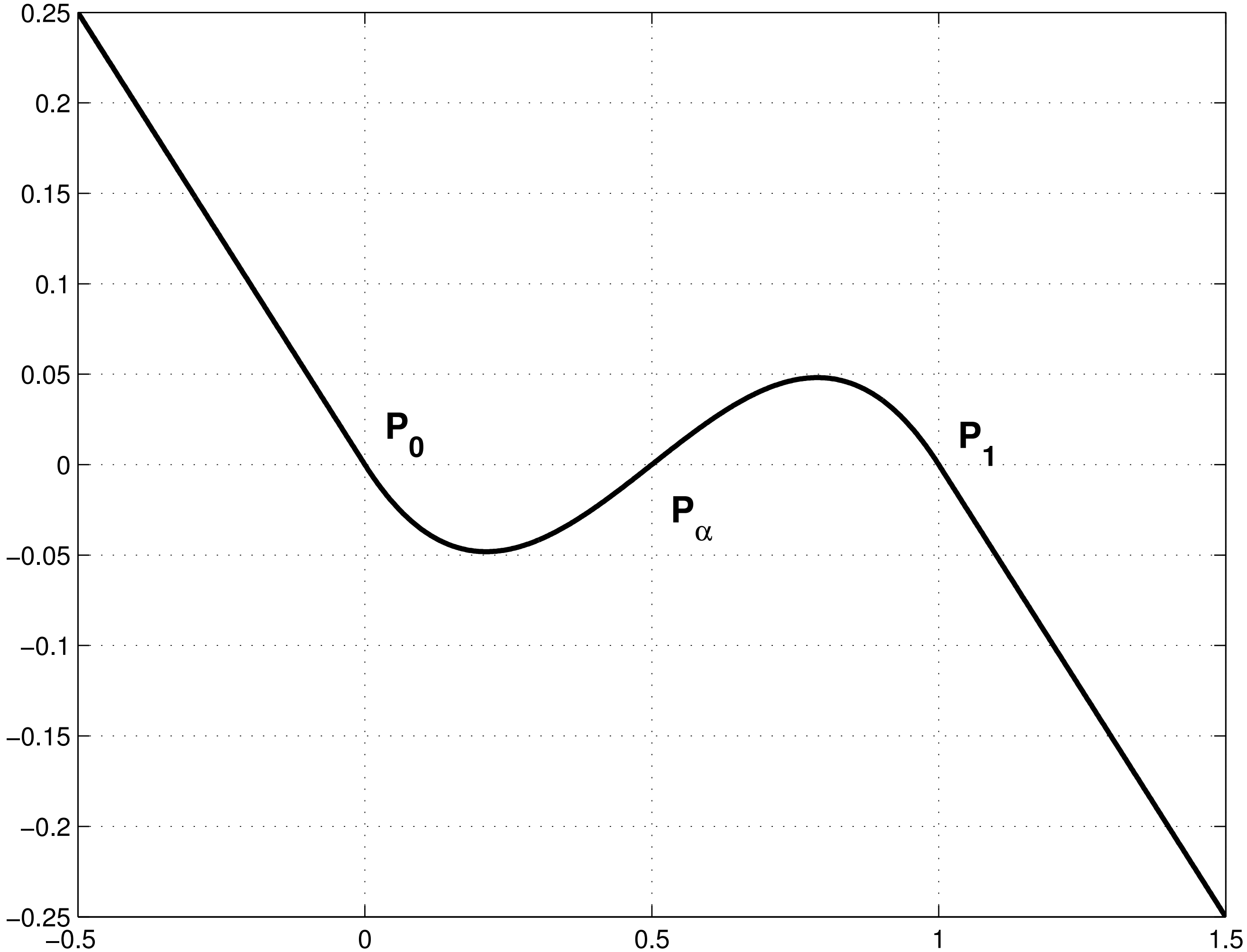


$\alpha = a_{11} - 1, \beta = \sqrt{|a_{12}a_{21}|}$ with $|a_{12}| < \alpha$ and $|a_{21}| > \alpha$

Equilibrium point bifurcations: *Nagumo* Equation

$$\frac{\partial \tilde{x}}{\partial t} = D\frac{\partial^2 \tilde{x}}{\partial z^2} + f(\tilde{x}) \qquad (0 \le z \le l_z) \quad \frac{\partial \tilde{x}}{\partial z} = 0\,|_{z=0,l_z}$$

$$f(\tilde{x}) = \begin{cases} -\alpha\tilde{x} & \tilde{x} \le 0 \\ -\tilde{x}^3 + (1+\alpha)\tilde{x}^2 - \alpha\tilde{x} & 0 \le \tilde{x} \le 1 \\ (\alpha - 1)(\tilde{x} - 1) & \tilde{x} \ge 1 \end{cases}$$


0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.5
0
0.5
1
1.5
P0
Pα
P1

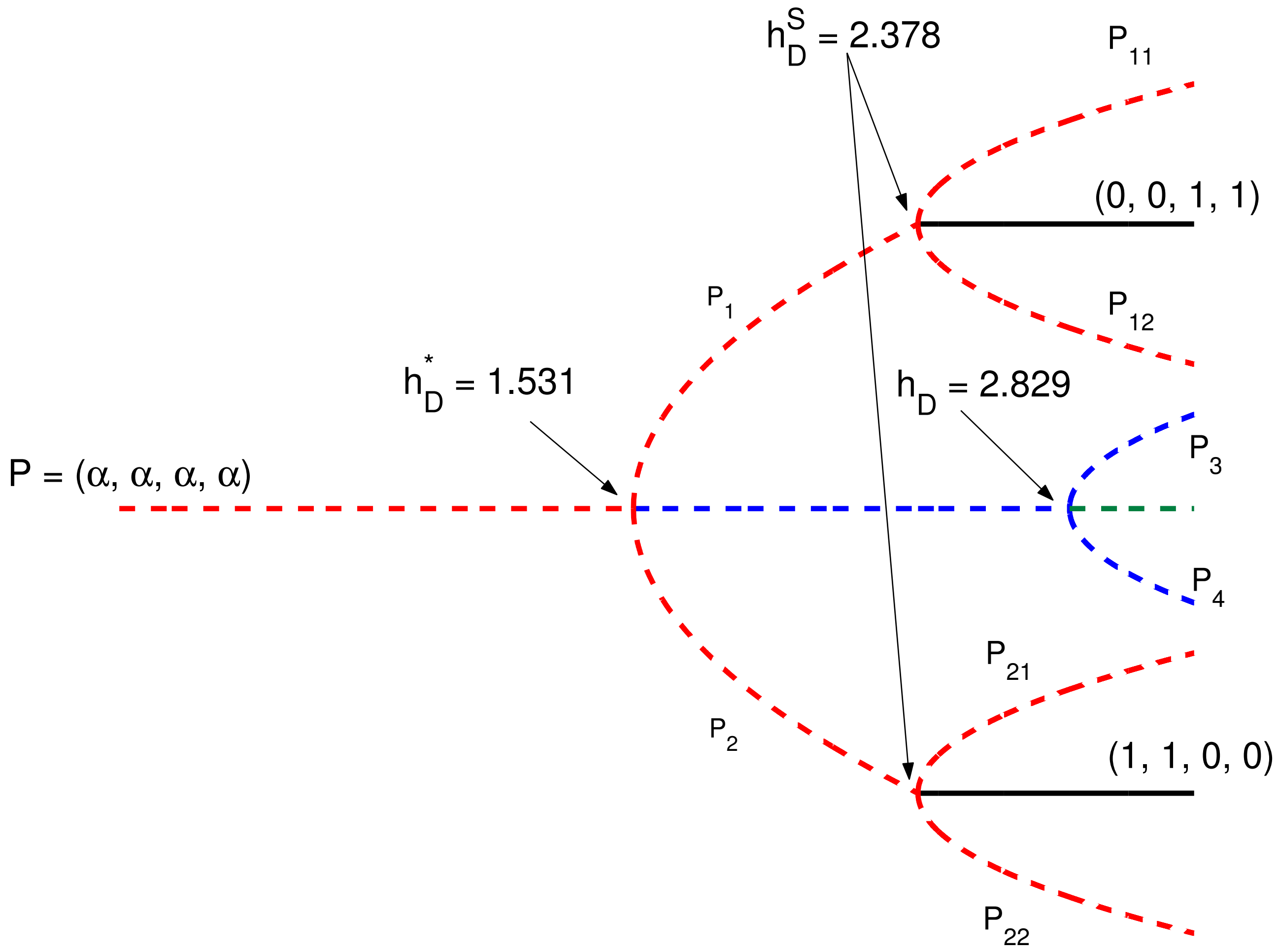
$$D\,\frac{\partial^2 \tilde{x}}{\partial z^2} \approx \frac{D}{h^2}\,[x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)]$$

$$\Downarrow$$

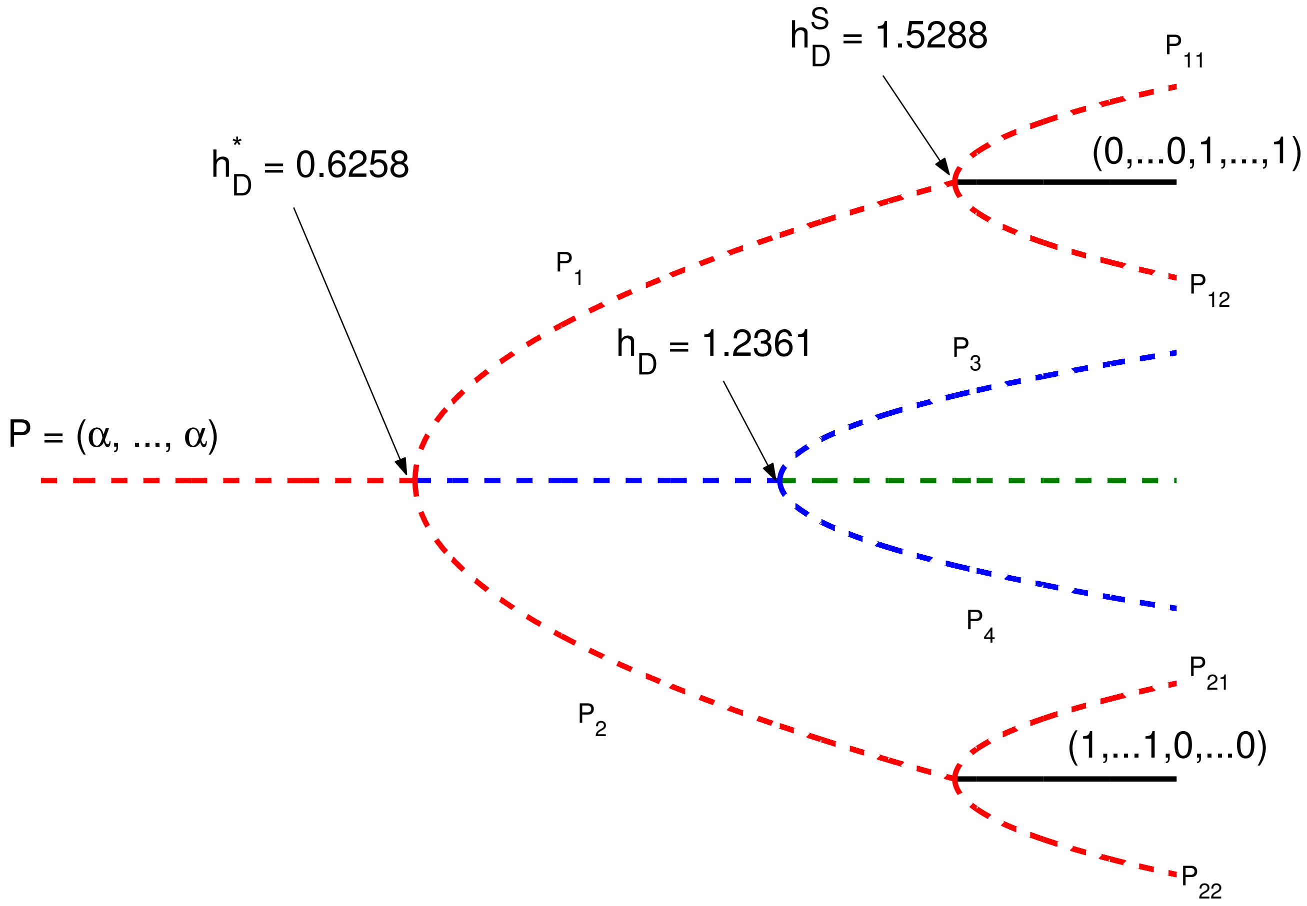
$$\begin{cases} \frac{dx_1(t)}{dt} & = & \frac{D}{h^2}\,[x_2(t) - x_1(t)] + f[x_1(t)] \\ \frac{dx_i(t)}{dt} & = & \frac{D}{h^2}\,[x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)] + f[x_i(t)] \quad (i \neq 2,\ N) \\ \frac{dx_N(t)}{dt} & = & \frac{D}{h^2}\,[x_{N-1}(t) - x_N(t)] + f[x_N(t)] \end{cases}$$

Normalized discretization step $h_D = \frac{h}{\sqrt{D}}$

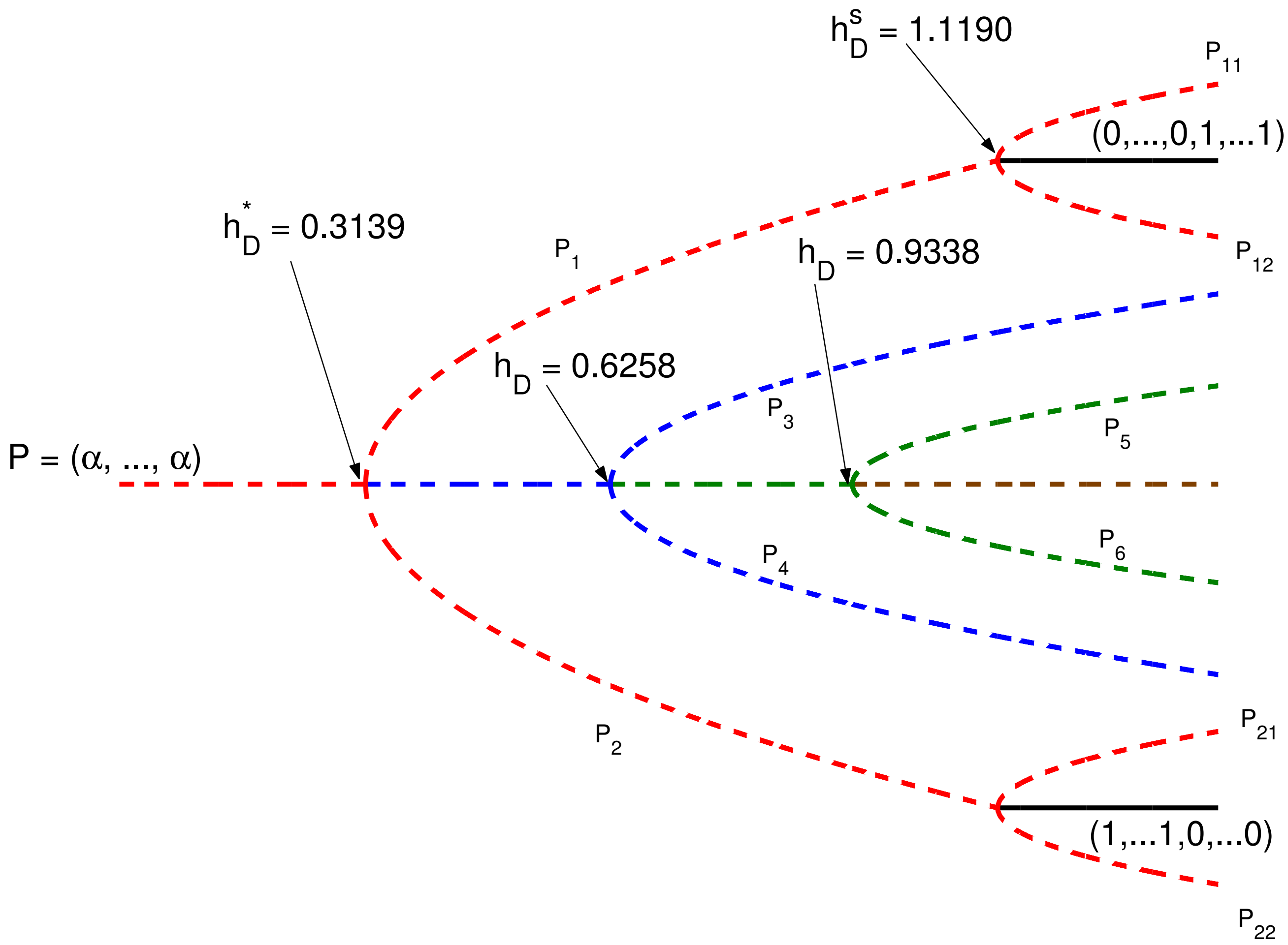
1. For any number N of cells there exists a normalized discretization step h_D^* such that for $0 \leq h_D < h_D^*$ the CNN equation presents only three equilibrium points:
   - Two stable equilibrium points $P_a = (0, 0, ..., 0, 0)$ and $P_b = (1, 1, ..., 1, 1)$, whose Jacobian matrix exhibits N negative real eigenvalues.
   - One unstable equilibrium point $P = (\alpha, \alpha, ..., \alpha, \alpha)$ that presents one positive real eigenvalue.
2. For each finite values of $h_D < h_D^*$ the CNN equation and the PDE models are *topologically equivalent*.
3. If $h_D \geq h_D^*$, then the CNN equation still presents the two stable equilibrium points P_a and P_b.
   The unstable point $P = (\alpha, \alpha, ..., \alpha, \alpha)$ undergoes a series of pitchfork bifurcations, that finally gives rise to the emergence of a number of additional stable equilibrium points.


$h_D^S = 2.378$
P_{11}
(0, 0, 1, 1)
P_1
P_{12}
$h_D^* = 1.531$
$h_D = 2.829$
P = (α, α, α, α)
P_3
P_4
P_{21}
P_2
(1, 1, 0, 0)
P_{22}

Equilibrium point bifurcations in a 10 cell CNN (cont.)



Equilibrium point bifurcations in a 20 cell CNN (cont.)



4. The CNN equation is not *topologically equivalent* to the PDE model for $h_D \geq h_D^*$, i.e. after the occurrence of the first bifurcation.
5. For $h_D > h_D^s$ the discrete CNN equation presents a pair of stable equilibrium points that are not present in the original PDE, i.e. there is not a one-to-one correspondence between the attractors of the two models.

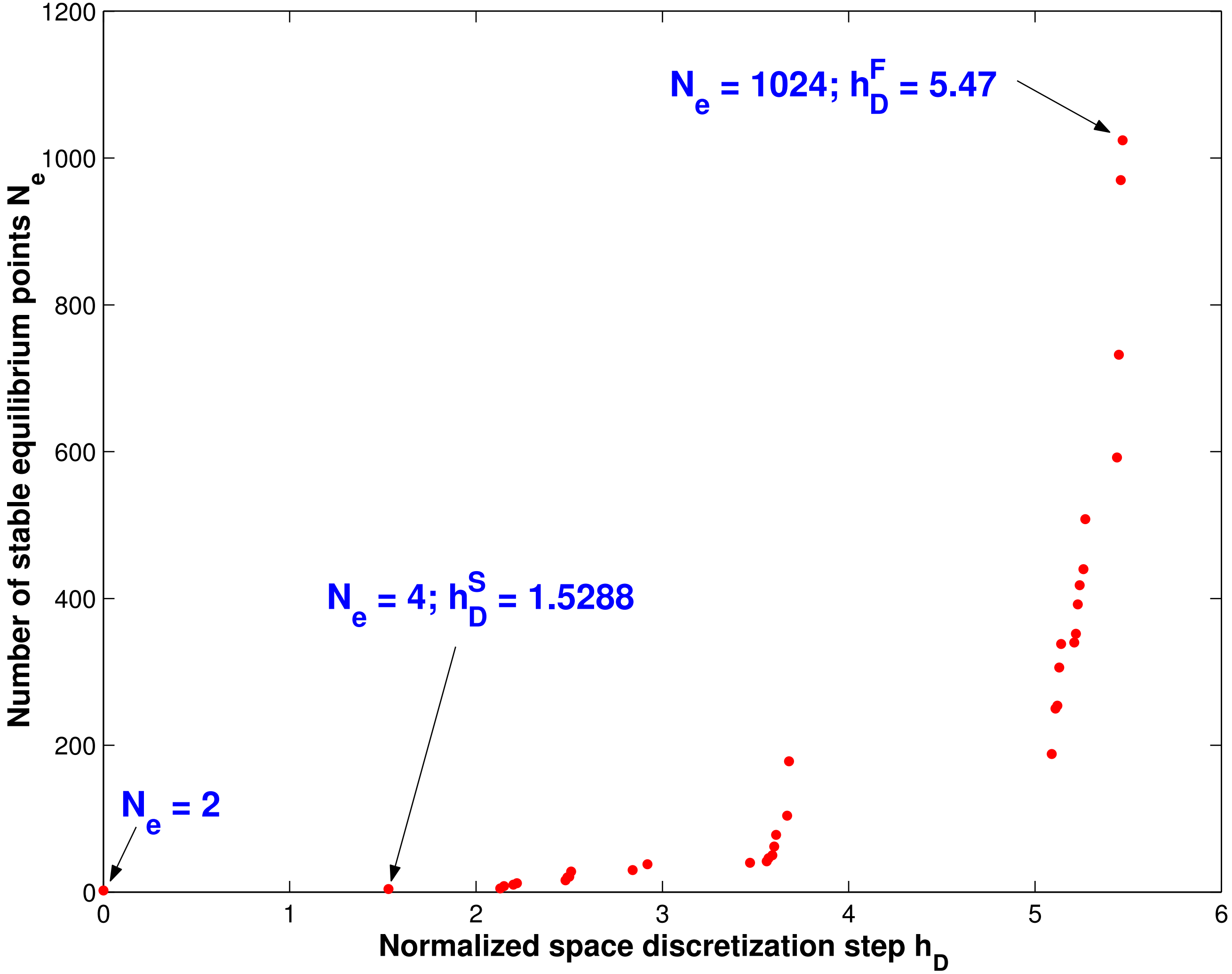
The propagation failure phenomena occurs for those $h_D > h_D^s$ such that there is not a one-to-one correspondence between the stable equilibrium points of the two models.

For each normalized discretization step h (even arbitrarily small) there exists a diffusion coefficient D such that $h_D = \frac{h}{\sqrt{D}} > h_S^D$,i.e. the CNN equation and the PDE model are not *topologically equivalent*.

6. For any number N of cells and diffusion coefficient D, there exists h_D^f such that for $h_D > h_D^f$ the cells can be considered uncoupled
   - Each uncoupled cell presents three equilibrium points:

$$x_i = \begin{cases} 0 & \text{stable} \\ 1 & \text{stable} \\ \alpha & \text{unstable} \end{cases}$$

   - The whole CNN presents 3^N equilibrium points:
     * 2^N stable points: $\{(1,0)\ (1,0),\ \ldots,\ (1,0),\ (1,0)\}$ (with N real negative eigenvalues).
     * $3^N - 2^N$ unstable points (with at least one positive real eigenvalue).


Number of stable equilibrium points N_e
Normalized space discretization step h_D
$N_e = 1024$; $h_D^F = 5.47$
$N_e = 4$; $h_D^S = 1.5288$
$N_e = 2$
0
200
400
600
800
1000
1200
1
2
3
4
5
6

Bifurcations and domains of attraction: Remarks

1. Local bifurcations can be studied by starting from a point of the parameter space that allows to easily compute all the equilibrium points.
   (a) CYCNNs (FRCNNs) with zero input and bias and dominant template. Each linear region presents an equilibrium point (either stable or instable).
   (b) Nagumo CNN with $h_D \to 0$.
2. Local bifurcations simply require the computation of the eigenvalues of the Jacobian matrix associated to each equilibrium point.
3. Global bifurcations require to determine the stable and unstable manifolds of the equilibrium points. Apart from simple network (plane systems) this is a formidable task.
4. The determination of the domains of attractions of the stable equilibrium points require to determine the unstable manifolds of the unstable equilibrium points. Apart from planar CNNs the task is practically impossible.

Existence of at least one stable equilibrium point

1. Several conditions for CYCNNs, based on the system description as a vector equation (*Tavsanoglu et al.*).

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) = -\boldsymbol{x} + \boldsymbol{A} + \boldsymbol{y} + \boldsymbol{B}\boldsymbol{u} + \hat{\boldsymbol{I}}$$

   (a) They do not exploit neither local connectivity nor template space-invariance.

   (b) They cannot be expressed in term of the template elements.

2. Some conditions directly exploits local connectivity and space-invariance (*Gilli et al.*)

3. Coexistence of stable equilibrium points and stable limit cycles has been observed (*Takahashi et al.*). The existence of stable equilibrium points does not imply stability.

Dynamic behavior: Stability

- Complete stability (Convergence): each trajectory converges towards an equilibrium point.
  - The system in general presents several equilibrium points.
  - All the attractors are stable equilibrium points.
- Stability (Convergence) almost everywhere: each trajectory (with the exception of a set of measure zero) converge towards an equilibrium point.
  - The systems may admits unstable periodic/non-periodic invariant limit sets, e.g. unstable limit cycles.
  - All the attractors are stable equilibrium points.
- Global stability: each trajectory converges towards a single stable equilibrium point, that is globally asymptotically stable.

- Lyapunov function $V(x)$ with non-positive time derivative along the system trajectories, i.e. $\dot{V}(x) \leq 0$
- All the trajectories are bounded.
- The equilibrium points are isolated.

$$\Downarrow$$

La Salle principle ensures the Complete stability

$\ast$ CYCNNs with symmetric templates (*Chua et al.*)

$\ast$ CYCNNs with diagonal dominant and related templates (*Takahasci et al.*)

$\ast$ CYCNNs with a class of non-positive and non-symmetric templates (*Gilli et al.*)

Stability almost everywhere

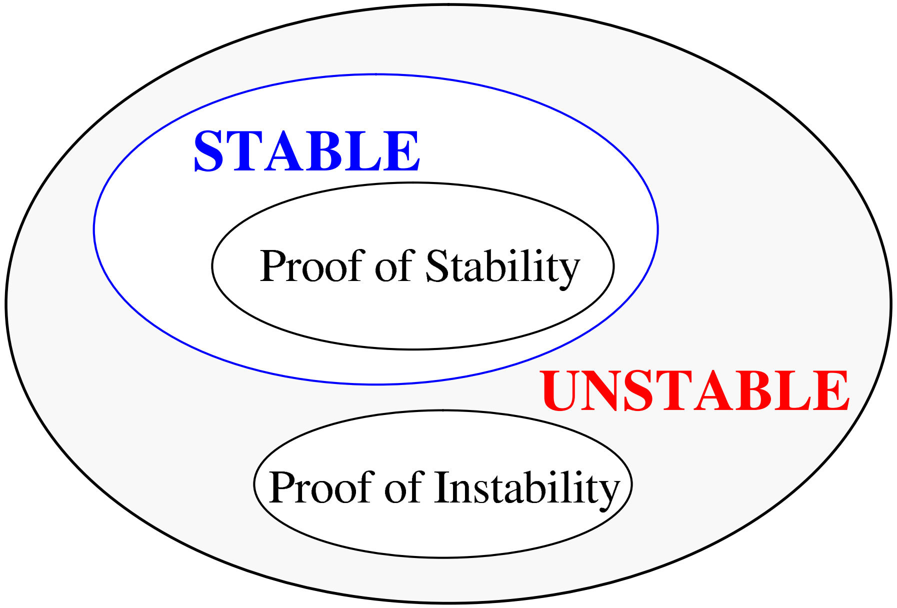
Based on some important results of Hirsch on cooperative systems

$$\Downarrow$$

- $*$ Irreducible CYCNNs with positive templates (*Chua et al.*)
- $*$ Irreducible CYCNNs with templates that can be mapped onto positive templates, through diagonal state transformations (*Wu et al.*)

Global stability

- $*$ Several results related to general neural networks.
- $*$ CYCNNs with a single equilibrium point are not suitable for signal and information processing.



*Most stability conditions do not exploit local connectivity and template space-invariance. The conditions under which stability can be proved are identical for both CYCNNs and FRCNNs (*Gilli et al.)

Dynamic behavior: Periodic limit cycles

1. Determination of all the periodic limit cycles (either stable and unstable) and their Floquet's multipliers (FMs)
   (a) Stable: one FM is zero; the absolute values of all the other FMs is less than 1.
   (b) Unstable: one FM is zero; at least one FM has absolute value grater than 1.
2. Limit cycle bifurcations
   (a) Tangent (fold) bifurcation: one FM equals 1.
   (b) Period doubling (flip) bifurcation: one FM equals -1.
   (c) Naimark Sacker bifurcation: the module of a pair of complex FMs is 1.
3. Estimation of the domains of attractions.

1. In large scale dynamical systems, like CNNs, the sole simulation does not allow to identify all the limit cycles (either stable or unstable).
   (a) It would require to consider *infinitely many* initial conditions.
   (b) Unstable limit cycles cannot be detected through simulation.
2. Through spectral techniques, the computation of all the limit cycles is reduced to a non-differential (sometimes algebraic) problem.
   (a) Describing function technique.
   (b) Harmonic balance technique.

Each one of the $N \times M$ state variables is represented through a single harmonic (with given amplitude, frequency and phase) and a bias

$$x_{ij}(t) \approx A_{ij} + B_{ij} \sin\left(\omega t + \eta_{ij}\right) \qquad \eta_{11} = 0$$

$$f\left(x_{ij}(t)\right) \approx F_{ij}^{A} + F_{ij}^{B} \sin\left(\omega t + \eta_{ij}\right)$$

$$F_{ij}^{A} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f[A_{ij} + B_{ij} \sin(\theta)] d\theta$$

$$F_{ij}^{B} = \frac{1}{\pi} \int_{-\pi}^{\pi} f[A_{ij} + B_{ij} \sin(\theta)] \sin(\theta) d\theta$$

1. The substitution into the state equation reduces the problem of detecting all the limit cycles to that of finding all the solutions of a set of $3(N \times M)$ equations, with the following unknowns
   (a) the frequency ω.
   (b) the $N \times M$ bias amplitudes A_{ij}.
   (c) the $N \times M$ harmonic amplitudes B_{ij}.
   (d) the $N \times M - 1$ harmonic phases η_{ij} (since η_{11} is assumed to be zero).
2. The conditions under which the describing function technique yields rigorous results are rather restrictive (*Mees et al.*). However in most cases it works (*Gilli et al.*).
3. Limit cycle stability and bifurcations can be studied through approximate methods based on the describing function technique (see *Genesio* and *Gilli* for the extension to large scale dynamical systems).

1. The characteristics of the limit cycles identified through the describing function technique can be investigated through the harmonic balance technique.

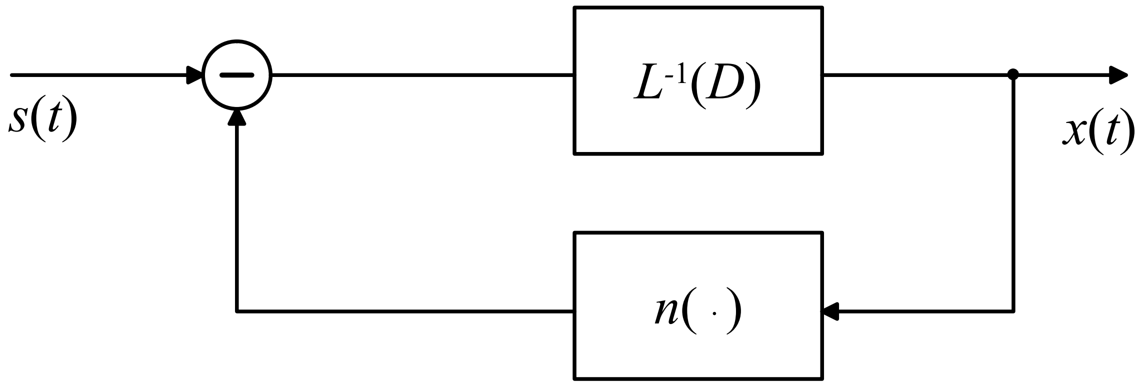
2. Each state variable is represented by means of a finite number H of harmonics:

$$x_{ij}(t) = A_{ij}^0 + \sum_{k=1}^{H} A_{ij}^k \cos(k\omega t) + B_{ij}^k \sin(k\omega t) \qquad A_{11}^1 = 0$$

Set of $(2H+1) \times N \times M$ unknowns

- the frequency ω.
- the $N \times M$ bias amplitudes A_{ij}^0.
- the $2H \times N \times M - 1$ harmonic amplitudes A_{ij}^k and B_{ij}^k (A_{11}^1 is assumed to be zero).

The HB technique for scalar Lur'e systems



$$L(D)x(t) + \boldsymbol{n}[x(t)] = s(t), \qquad x(t) \in R$$

If the systems admits of a periodic solution of period T, then $x(t)$ can be expanded through the Fourier series

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega t) + B_k \sin(k\omega t) \qquad \omega = \frac{2\pi}{T}$$

Examples

Third order oscillator

$$L(D) = \frac{D^3 + (1+\alpha)D^2 + \beta D + \alpha\beta}{\alpha\left(D^2 + D + \beta\right)} \qquad \boldsymbol{n}(x) = -\frac{8}{7}x + \frac{4}{63}x^3$$

Second order oscillator

$$L(D) = \frac{LCD^2 - LD + 1}{kLD} \qquad \boldsymbol{n}(x) = x^3$$

The harmonic balance (HB) technique

1. The state is represented through a **finite** (**N**) number of harmonics

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(k\omega t) + B_k \sin(k\omega t)$$

2. The term $L(D)x(t)$ yields:

$$\begin{aligned} L(D)x(t) \quad &= \quad L(0)A_0 \quad + \quad \sum_{k=1}^{N} \{\boldsymbol{Re}[L(jk\omega)]A_k + \boldsymbol{Im}[L(jk\omega)]B_k\} \cos(k\omega t) \\ & \qquad\qquad\qquad + \quad \sum_{k=1}^{N} \{\boldsymbol{Re}[L(jk\omega)]B_k - \boldsymbol{Im}[L(jk\omega)A_k\} \sin(k\omega t) \end{aligned}$$

3. The term $\boldsymbol{n}[x(t)]$ yields (by truncating the series to N harmonics):

$$\boldsymbol{n}[x(t)] = C_0 + \sum_{k=1}^{N} C_k \cos(k\omega t) + D_k \sin(k\omega t)$$

$$\begin{aligned}
C_0 &= \frac{1}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \sin(k\omega t) + B_k \cos(k\omega t)\right]\ dt \\
C_k &= \frac{2}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \sin(k\omega t) + B_k \cos(k\omega t)\right]\ \cos(k\omega t)\ dt \\
D_k &= \frac{2}{T}\int_0^T \boldsymbol{n}\left[A_0 + \sum_{k=1}^{N} A_k \sin(k\omega t) + B_k \cos(k\omega t)\right]\ \sin(k\omega t)\ dt
\end{aligned}$$

3. The term $s(t)$ yields (by truncating the series to N harmonics):

$$s(t) = P_0 + \sum_{k=1}^{N} P_k \cos(k\omega t) + Q_k \sin(k\omega t)$$

$$\begin{aligned} P_0 &= \frac{1}{T}\int_0^T s(t)\, dt \\ P_k &= \frac{2}{T}\int_0^T s(t)\cos(k\omega t)\, dt \\ Q_k &= \frac{2}{T}\int_0^T s(t)\ \sin(k\omega t)\, dt \end{aligned}$$

4. A set of $2N+1$ **nonlinear equations** is obtained, by equating the coefficients of the constant term and of the harmonics $\cos(k\omega t)$, $\sin(k\omega t)$

$$\begin{array}{lcccll}
L(0)A_0 & + & C_0(A_0,...,B_N) & = & P_0 & \\
\boldsymbol{Re}[L(jk\omega)]A_k - \boldsymbol{Im}[L(jk\omega)]B_k & + & C_k(A_0,...,B_N) & = & P_k & 1 \le k \le N \\
\boldsymbol{Im}[L(jk\omega)]A_k + \boldsymbol{Re}[L(jk\omega)]B_k & + & D_k(A_0,...,B_N) & = & Q_k & 1 \le k \le N
\end{array}$$

5. **Autonomous systems**: the term A_1 is assumed to be equal to zero (i.e. the phase of the first harmonic of $x(t)$ is arbitrarily fixed); since ω is unknown, the system has an equal number $[(2N+1)]$ of equations and unknowns.

Efficient HB implementations

1. Consider the time samples vectors

$$y(t) = L(D)x(t)$$

$$\underline{y} = [y(t_0),\ y(t_1), ..., y(t_{2N}),\ y(t_{2N+1})]'$$

$$\underline{x} = [x(t_0),\ x(t_1), ..., x(t_{2N}),\ x(t_{2N+1})]'$$

$$\underline{s} = [s(t_0),\ s(t_1), ..., s(t_{2N}),\ s(t_{2N+1})]'$$

$$t_p = \frac{T}{2N+1}p \qquad\qquad p = 1, \ldots, 2N+1$$

2. Impose that the HB equation be satisfied for $t = t_p$

$$y(t_p) + \boldsymbol{n}[x(t_p)] = s(t_p), \qquad p = 1, \ldots, 2N+1$$

that in vector notation yields

$$\underline{y} + \boldsymbol{n}[\underline{x}] = \underline{s}$$

with

$$\boldsymbol{n}[\underline{x}] = [\ \boldsymbol{n}[x(t_1)],\ \boldsymbol{n}[x(t_2)], ..., \boldsymbol{n}[x(t_{2N+1})]\ ]'$$

$$\underline{x} = \Gamma^{-1}\underline{X}, \qquad \underline{X} = [A_0,\ A_1, ..., A_N,\ B_1, ..., B_N]'$$

$$\Gamma^{-1} = \begin{bmatrix} 1 & \gamma^c_{1,1} & \gamma^s_{1,1} & \cdots & \gamma^c_{1,N} & \gamma^s_{1,N} \\ 1 & \gamma^c_{2,1} & \gamma^s_{2,1} & \cdots & \gamma^c_{2,N} & \gamma^s_{2,N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \gamma^c_{2N+1,1} & \gamma^s_{2N+1,1} & \cdots & \gamma^c_{2N+1,N} & \gamma^s_{2N+1,N} \end{bmatrix}$$

$$\begin{aligned} \gamma^c_{p,q} &= \cos(q\omega t_p) = \cos\left(\frac{q2\pi p}{2N+1}\right) \\ \gamma^s_{p,q} &= \sin(q\omega t_p) = \sin\left(\frac{q2\pi p}{2N+1}\right) \end{aligned}$$

$$\underline{y} = \Gamma^{-1}\,\Omega(\omega)\,\underline{X}$$

$$\Omega(\omega) = \begin{bmatrix} L(0) & 0 & 0 & \dots & 0 & 0 \\ 0 & R_1 & I_1 & \dots & 0 & 0 \\ 0 & -I_1 & R_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & R_N & I_N \\ 0 & 0 & 0 & \dots & -I_N & R_N \end{bmatrix}$$

$$R_k = \boldsymbol{Re}\{L(jk\omega)\}, \quad I_k = \boldsymbol{Im}\{L(jk\omega)\}$$

$$\boldsymbol{n}[\underline{x}] = \boldsymbol{n}[\Gamma^{-1}\underline{X}]$$

$$\underline{y} + \boldsymbol{n}[\underline{x}] = \underline{s}$$

$$\Downarrow$$

$$\Gamma^{-1}\ \Omega(\omega)\ X + \boldsymbol{n}[\Gamma^{-1}\underline{X}] = \underline{s}$$

$$\Downarrow$$

$$\Omega(\omega)\ \underline{X} + \Gamma\ \boldsymbol{n}[\Gamma^{-1}\underline{X}] = \Gamma\ \underline{s}$$

The $2N + 1$ equations in the $2N + 1$ unknowns $\underline{X}$ can be solved without performing any integrals.

Limit cycle stability and bifurcations

Variational equation: $x(t) = \overline{x}(t) + \tilde{x}(t)$

$$L(D)\tilde{x}(t) + g(t)\tilde{x}(t) = 0$$

$$g(t) = \left.\frac{\mathrm{d}n(\zeta)}{\mathrm{d}\zeta}\right|_{\zeta=\overline{x}(t)}$$

$$\tilde{x}(t) = \sum_{i=1}^{D} H_i v_i(t) \exp(\lambda_i t) \quad (D \;\rightarrow\; \text{system dimension})$$

Floquet's multipliers $\rightarrow \exp(\lambda_i T)$

Periodic limit cycle: final remarks

1. The describing function technique is very effective for detecting the existence of limit cycles (either stable or unstable) and also for a preliminary study of their bifurcations.

2. The harmonic balance technique allows to determine with a good accuracy the main limit cycle characteristics.

3. HB based technique can be exploited for computing FMs, even in large-scale systems (*Gilli et al.*).

4. Once the limit cycle has been detected through a spectral technique, the Floquet's multipliers can also be computed via a time-domain technique. The application to large arrays of nonlinear oscillators (Chua's circuits) has allowed to determine all the significant limit cycle bifurcations (*Gilli et al.*).

A CNN can also be considered as a dynamical system with a time and space dependence. The extension of the describing function and of the harmonic balance technique to space-time systems allows to detect important space-time phenomena. (*Gilli et al.*)

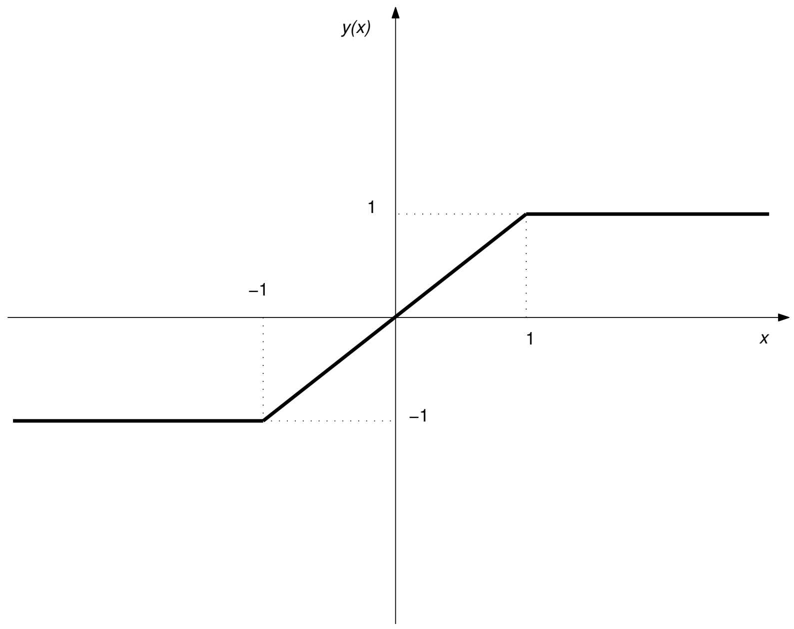
Dynamic behavior: non-periodic attractors

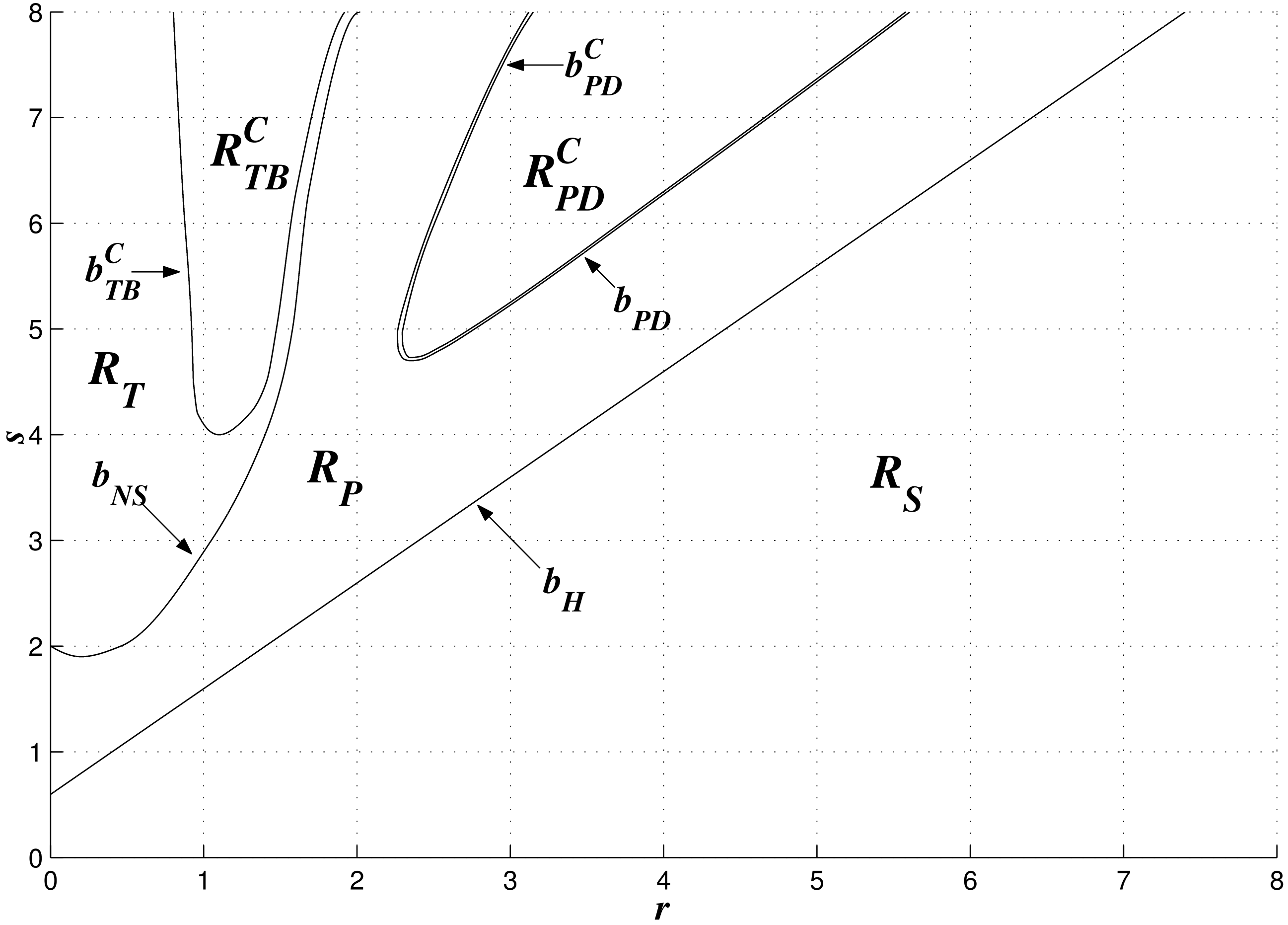
Non-periodic attractors can be detected through the following tools:

1. Bifurcation analysis.
   (a) Neimark Sacker: torus.
   (b) Sequence of period doubling bifurcations: period doubling route to chaos.
   (c) Sequence of Neimark Sacker bifurcations: torus breakdown route to chaos.
2. Computation of the Lyapunov exponents.
   (a) Torus attractor: more than one exponent equals zero.
   (b) Chaotic attractor: at least one positive exponent.
3. Evaluation of the power spectrum.

Example of complex dynamics in CYCNN

$$T^A = \begin{bmatrix} 0 & r & 0 \\ s & p & -s \\ 0 & r & 0 \end{bmatrix} \quad \text{with } p > 1,\ r > 0,\ s > 0$$





R^C_{TB}
R^C_{PD}
b^C_{PD}
b^C_{TB}
b_{PD}
R_T
R_P
R_S
b_{NS}
b_H
s
r
0
1
2
3
4
5
6
7
8

Conclusions

1. *CNN are large-scale nonlinear dynamical systems with local connectivity and space-invariant structure.*
2. *In order to develop a rigorous design method a satisfactory knowledge of CNN dynamics is necessary.*
3. *The analysis of the dynamics requires the introduction of new analytical and numerical techniques.*
4. *Some results based on classical and new techniques and some open problems have been presented.*